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Mass transport in a parabolic conduit

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Abstract—The local concentration is determined for a viscous liquid flowing through a parabolic conduit. The concentration at the parabolic wall is considered constant, as is the initial concentration at the inlet. The liquid flow is considered as creeping flow and its velocity distribution is determined by solving the biharmonic equation of the streamfunction. The local concentration is evaluated numerically from the analytical results for various parabolic conduits.

1. INTRODUCTION

FOR THE past century the heat and mass transport problem with the inclusion of convection has received considerable attention. The transport of heat or concentration of a material in a liquid has been investigated in conduits and tubes for plug and laminar flow by Graetz, Nusselt and others [1–6]. In all these investigations the geometry of the system was such, that the cross-section of the conduit or tube did not change. The partial differential equation describing the problem could be separated and the remaining ordinary differential equation with usually varying coefficients had to be solved by some numerical method, which usually was performed by applying the Runge–Kutta procedure. In some cases an exact analytical solution could be obtained by proper transformation of the ordinary differential equation into the confluent hypergeometric equation. In other cases the Galerkin procedure leads to very good approximations. This method was used by the author for the treatment of the problem for a non-Newtonian liquid of the Ostwald–de-Waele type [7–9]. In some configurations, however, the cross-sectional area changes, such as in a diverging straight line conduit or a conical tube. These cases were also treated by the author [10, 11]. The following investigation treats the mass transport in a conduit, in which the cross-sectional area changes according to parabolic walls, i.e. the walls are not represented by straight lines, but by parabolas, at which the wall concentration c_w is considered constant. Viscous creeping flow will be used flowing in the ξ-direction and exhibiting only the velocity component $u(\xi, \eta)$ in this direction.

2. BASIC EQUATIONS AND SOLUTION

For the determination of the local concentration of a component in a moving viscous liquid in a conduit or tube with a completely developed velocity profile, the following second-order partial differential equation has to be solved:

$$D\Delta c - \mathbf{v}\cdot\operatorname{grad} c = 0 \tag{1}$$

where D is the diffusion coefficient, c the local concentration and $\mathbf{v}$ the velocity of the liquid. For a parabolic conduit, i.e. a system with a parabolically changing cross-section the equation, that has to be solved is given by

$$\frac{D}{(\xi^2+\eta^2)}\frac{\partial^2 c}{\partial \eta^2} - \frac{u(\xi,\eta)}{\sqrt{(\xi^2+\eta^2)}}\frac{\partial c}{\partial \xi} = 0 \tag{2}$$

where molecular diffusion in the flow direction ξ has been neglected in comparison to the convectional part. This partial differential equation has to be solved with the boundary condition

$$c = c_w \text{ at the wall of the conduit } \eta = \eta_0 \tag{3a}$$

and the initial condition

$$c = c_i \text{ at the inlet } \xi = \xi_0 \tag{3b}$$

where c_w and c_i are constants. The parabolic coordinate ξ and η relate to the Cartesian coordinates x and y by

$$x = \xi\eta, \quad y = \tfrac{1}{2}(\xi^2-\eta^2)$$

and the lines ξ = const. and η = const. both represent parabolas, which are orthogonal to each other (Fig. 1).

2.1. *The flow problem*

The viscous flow of a homogeneous and incompressible Newtonian liquid in a parabolic conduit is treated, for which $\eta = \eta_0$ is chosen as its wall. Since the liquid adheres to the wall, $\eta = \eta_0$ is a streamline $\Psi(\eta_0)$, indicating, that its normal velocity vanishes. Ψ is the streamfunction. In addition the tangential velocity $\partial\Psi/\partial\eta$ should vanish at the wall $\eta = \eta_0$. Therefore, it is

$$\frac{\partial\Psi}{\partial\eta} = 0 \quad \text{at the wall } \eta = \eta_0. \tag{4}$$

From the continuity equation and streamfunction one obtains

NOMENCLATURE

c	concentration
c_w	concentration at the wall of the conduit
c_i	concentration at the inlet
D	diffusion coefficient
${}_1F_1$	confluent hypergeometric function
u	flow velocity of viscous liquid in the ξ-direction
$\dot{V}_0$	volumetric flow.

Greek symbols

γ_n	eigenvalues, $\gamma^2 = 3\dot{V}_0\lambda^2\eta_0/4D$
η_0	wall of parabolic conduit
ξ, η	parabolic coordinates
ξ_0	inlet of parabolic conduit
Ψ	streamfunction.

$$u = \frac{\partial\Psi/\partial\eta}{\sqrt{(\xi^2+\eta^2)}} \tag{5}$$

where Ψ is only a function of η, i.e. $\Psi = \Psi(\eta)$. It is $\mathbf{v} = u\mathbf{e}_\xi + v\mathbf{e}_\eta$, with $v = 0$.

The axis of the conduit $\eta = 0$ is also a streamline expressing

$$\Psi(0) = 0 \quad \text{for } \eta = 0. \tag{6}$$

Linearized (creeping) flow thus requires the solution of the biharmonic equation of the streamfunction

$$\bar{\Delta}^2\Psi = 0 \tag{7}$$

with the above boundary conditions (4) and (6), as well as the volumetric flow condition

$$\dot{V}_0 = 2\int_0^{\eta_0} \sqrt{(\xi^2+\eta^2)}\,u(\xi,\eta)\,\mathrm{d}\eta. \tag{8}$$

The operator $\bar{\Delta}$ is given by

$$\bar{\Delta} \equiv \frac{1}{(\xi^2+\eta^2)}\left[\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2}\right]$$

and yields applied again on

$$\bar{\Delta}\Psi = \frac{\mathrm{d}^2\Psi/\mathrm{d}\eta^2}{(\xi^2+\eta^2)}$$

the expression

$$\bar{\Delta}^2\Psi = \frac{1}{(\xi^2+\eta^2)}\left[\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2}\right]\frac{\mathrm{d}^2\Psi/\mathrm{d}\eta^2}{(\xi^2+\eta^2)} = 0. \tag{9}$$

Performing the necessary evaluation of this equation and observing the fact that the streamfunction is only a function of the coordinate η renders as coefficients of ξ^4, ξ^2 and the ξ-independent term that ordinary differential equations for Ψ which has to be satisfied simultaneously. They are

$$\frac{\mathrm{d}^4\Psi}{\mathrm{d}\eta^4} = 0, \quad \eta^2\frac{\mathrm{d}^4\Psi}{\mathrm{d}\eta^4} - 2\eta\frac{\mathrm{d}^3\Psi}{\mathrm{d}\eta^3} + 2\frac{\mathrm{d}^2\Psi}{\mathrm{d}\eta^2} = 0$$

and

$$\eta^2\frac{\mathrm{d}^4\Psi}{\mathrm{d}\eta^4} - 4\eta\frac{\mathrm{d}^3\Psi}{\mathrm{d}\eta^3} + 4\frac{\mathrm{d}^2\Psi}{\mathrm{d}\eta^2} = 0.$$

The solution Ψ satisfying all three of them is given by

$$\Psi(\eta) = \tfrac{1}{6}A\eta^3 + B\eta + C \tag{10}$$

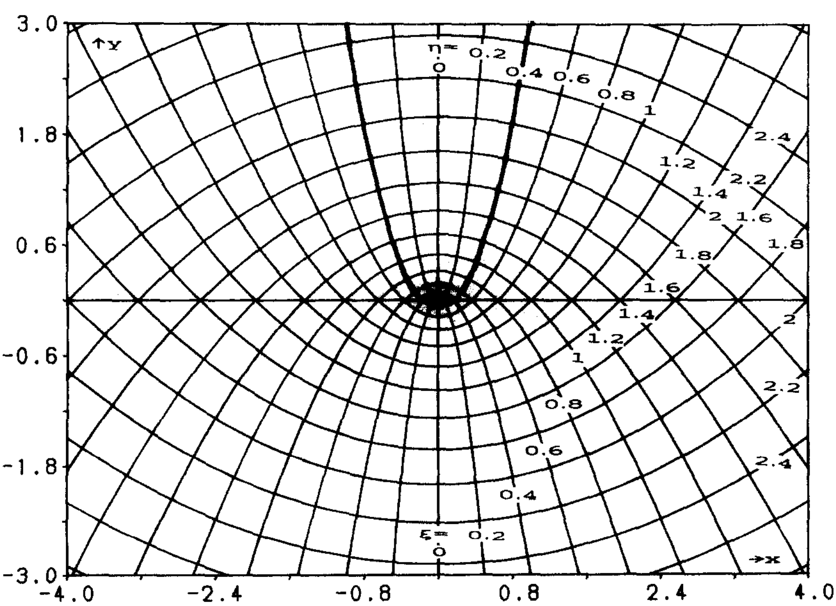


FIG. 1. Geometry and parabolic coordinates ξ, η.

which after introduction of conditions (4), (6) and (8) renders

$$\Psi(\eta) = \frac{\dot{V}_0}{4\eta_0^3}[3\eta_0^2\eta - \eta^3]. \tag{11}$$

The velocity distribution is therefore given with equation (5) as

$$u(\xi, \eta) = \frac{3\dot{V}_0}{4\eta_0^3}\frac{(\eta_0^2 - \eta^2)}{\sqrt{(\xi^2 + \eta^2)}} \tag{12}$$

and is presented in Fig. 2.

2.2. *The mass transport problem*

With the introduction of the velocity $u(\xi, \eta)$ into equation (2) we obtain the partial differential equation

$$\frac{\partial^2 c}{\partial \eta^2} - \frac{3\dot{V}_0}{4D\eta_0^3}(\eta_0^2 - \eta^2)\frac{\partial c}{\partial \xi} = 0 \tag{13}$$

for the determination of the local concentration $c(\xi, \eta)$. Transforming with

$$\frac{c - c_w}{c_i - c_w} \equiv \bar{C}(\xi, \eta) = C(\eta)\,e^{-\lambda^2(\xi - \xi_0)} \tag{14}$$

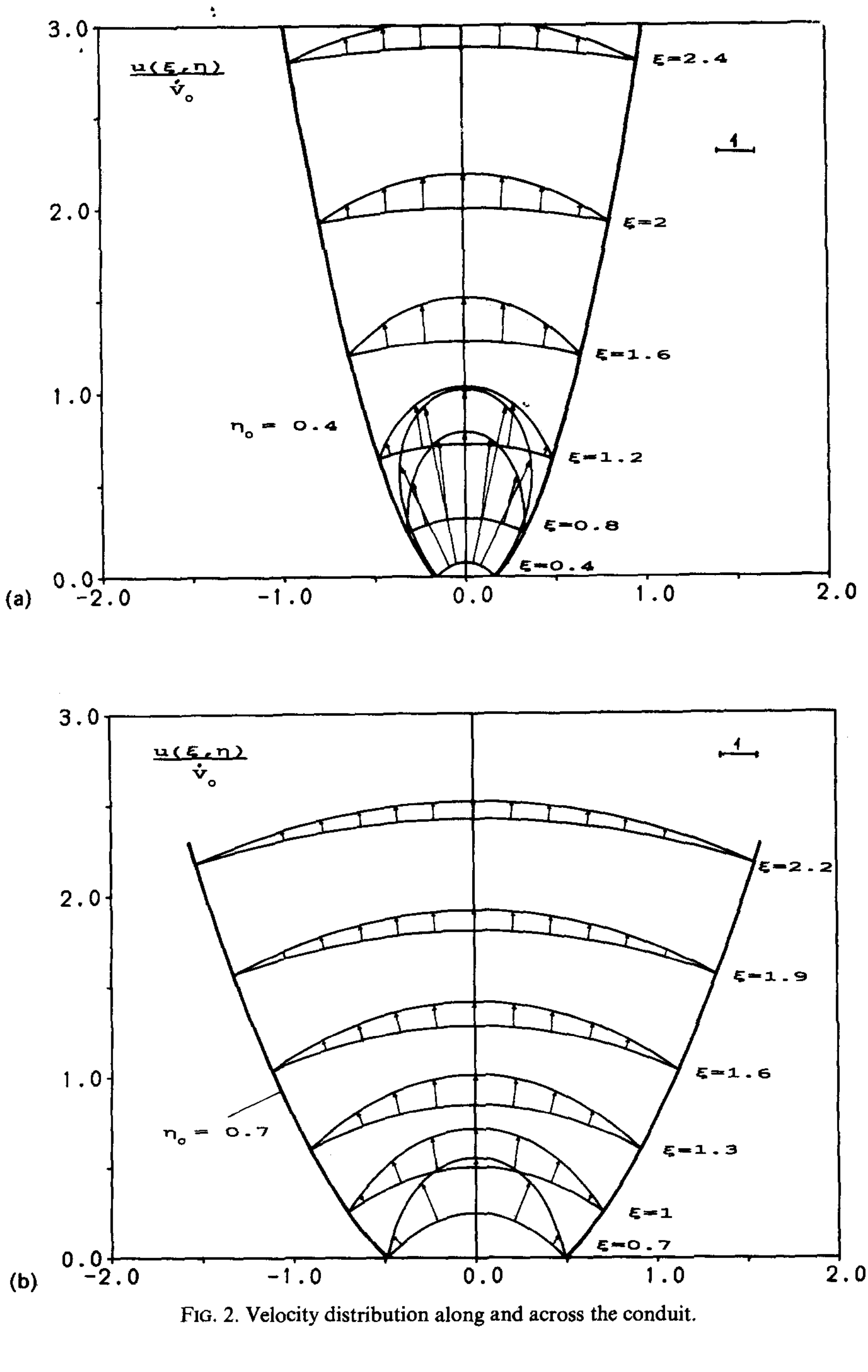
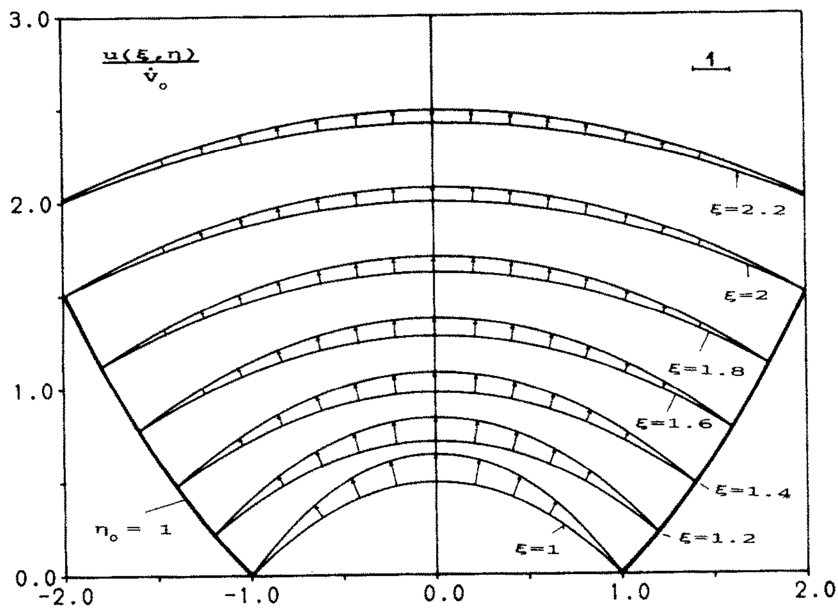


FIG. 2. Velocity distribution along and across the conduit.



(c)

FIG. 2.—*Continued.*

and introducing $\eta/\eta_0 = \zeta$ and $\gamma^2 \equiv 3\dot{V}_0\lambda^2\eta_0/4D$, one obtains the ordinary differential equation

$$\frac{d^2C}{d\zeta^2} + \gamma^2(1-\zeta^2)C = 0 \tag{15}$$

which after transformation with

$$\gamma\zeta^2 = z \quad \text{and} \quad C = e^{-z/2} f(z) \tag{16}$$

renders the confluent hypergeometric differential equation

$$z\frac{d^2f}{dz^2} + (\tfrac{1}{2}-z)\frac{df}{dz} - \tfrac{1}{4}(1-\gamma)f = 0 \tag{17}$$

exhibiting the solution (because of symmetry and the condition $dC/d\eta = 0$ at $\eta = 0$ only ${}_1F_1$ has been used)

$$f(z) = \alpha\, {}_1F_1(\tfrac{1}{4}(1-\gamma), \tfrac{1}{2}; z) \tag{18a}$$

where ${}_1F_1$ is the confluent hypergeometric series

$${}_1F_1(\alpha, \beta, z) = \sum_{\lambda=0}^{\infty} \frac{\Gamma(\beta)\Gamma(\alpha+\lambda)}{\Gamma(\alpha)\Gamma(\beta+\lambda)} \frac{z^\lambda}{\lambda!}. \tag{18b}$$

The concentration therefore will be given by

$$c(\xi,\eta) = c_w + (c_i - c_w) \sum_{n=1}^{\infty} E_n\, e^{-(\gamma_n/2)(\eta^2/\eta_0^2)}$$
$$\times\, {}_1F_1\left(\tfrac{1}{4}(1-\gamma_n), \tfrac{1}{2}; \frac{\eta^2}{\eta_0^2}\right) e^{-(4D_0/3\dot{V}_0\eta_0)\gamma_n^2(\xi-\xi_0)} \tag{19}$$

where γ_n are the eigenvalues, as obtained from condition (3a), i.e. $C(\eta_0) = 0$ or

$${}_1F_1(\tfrac{1}{4}(1-\gamma), \tfrac{1}{2}; \gamma) = 0 \quad \text{(Table 1)} \tag{20}$$

and where constants E_n have to be determined from initial condition (3b), i.e. $C(\eta) = 1$ at the inlet $\xi = \xi_0$. This yields

$$\sum_{n=1}^{\infty} E_n C_n(\zeta) = 1$$

$C_n(\zeta)$ being the function of $\eta/\eta_0 = \zeta$ in equation (19).

From equation (15) one obtains in the usual way the orthogonality relation of C_n, i.e.

$$\int_0^1 (1-\zeta^2)C_m(\zeta)C_n(\zeta)\,d\zeta$$

Table 1. Eigenvalues γ_n of the hypergeometric confluent function ${}_1F_1(\tfrac{1}{4}(1-\gamma_n), \tfrac{1}{2}; \gamma_n) = 0$

n	γ_n
1	1.6816
2	5.6699
3	9.6682
4	13.6677
5	17.6674
6	21.6672
7	25.6671
8	29.6670
9	33.6670
10	37.6669
11	41.6669
12	45.6669
13	49.6668
14	53.6668
15	57.6668
16	61.6668
17	65.6668
18	69.6668
19	73.6668
20	77.6668
21	81.6668
22	85.6668
23	89.6667
24	93.6667
25	97.6667

$$= \begin{cases} 0 & \text{for } m \neq n \\ \dfrac{1}{2\gamma_n}\dfrac{\partial C_n}{\partial \gamma_n}\dfrac{\partial C_n}{\partial \zeta}\Big|_{\zeta=1} & \text{for } m = n. \end{cases} \quad (21)$$

Constants E_n are therefore

$$E_n = \frac{\int_0^1 (1-\zeta^2)C_n(\zeta)\,d\zeta}{\int_0^1 (1-\zeta^2)C_n^2(\zeta)\,d\zeta}$$

which is with equation (21) and

$$\int_0^1 (1-\zeta^2)C_n(\zeta)\,d\zeta = -\frac{dC_n}{d\zeta}(1)/\gamma_n^2 \quad (22)$$

given by

$$E_n = -\frac{2}{\gamma_n \dfrac{dC_n}{d\gamma_n}\Big|_{\zeta=1}}. \quad (23)$$

Result (22) has been obtained from the integration of differential equation (15), observing that $dC_n/d\zeta(0) = 0$, i.e. symmetry of the concentration profile across the axis $\eta = 0$ of the conduit. The derivation of $\partial C_n/\partial \gamma_n$ yields with

$$C_n(\zeta) = e^{-(1/2)\gamma_n\zeta^2}\,{}_1F_1[\tfrac{1}{4}(1-\gamma_n), \tfrac{1}{2}; \gamma_n\zeta^2] \quad (24)$$

the expression

$$\frac{\partial C_n(J)}{\partial \gamma_n} = -\tfrac{1}{2}\zeta^2 C_n(\zeta) + e^{-(1/2)\gamma_n\zeta^2} \times \sum_{\nu=1}^{\infty} \frac{[\tfrac{1}{4}(1-\gamma_n)]_\nu(\gamma_n\zeta^2)^\nu}{(\tfrac{1}{2})_\nu \nu!}\left\{\frac{\nu}{\gamma_n} - \sum_{j=0}^{\nu-1}\frac{1}{(1-\gamma_n)+4j}\right\}. \quad (25)$$

With

$$\frac{d}{dz}[{}_1F_1(a,c,z)] = \frac{a}{c}\,{}_1F_1(a+1,c+1,z)$$

one obtains with equations (20) and (24)

$$\frac{\partial C_n}{\partial \zeta}(1) = e^{-\gamma_n/2}\gamma_n(1-\gamma_n)\,{}_1F_1(\tfrac{1}{4}(5-\gamma_n), \tfrac{3}{2}; \gamma_n). \quad (26)$$

With these results the orthogonality condition and the integration constants E_n are given. If the initial condition were not a constant, but a function of η the solution could be obtained in a similar way.

3. NUMERICAL EVALUATION AND CONCLUSIONS

The velocity distribution and the local concentration have been numerically evaluated for various parabolic conduits η_0 and at various coordinate lines ξ_j. In Figs. 2(a)–(c) the velocity distribution $u(\xi,\eta)/\dot{V}_0$ is presented in magnitude and direction. For a slowly increasing cross-section $\eta_0 = 0.4$ the vel-

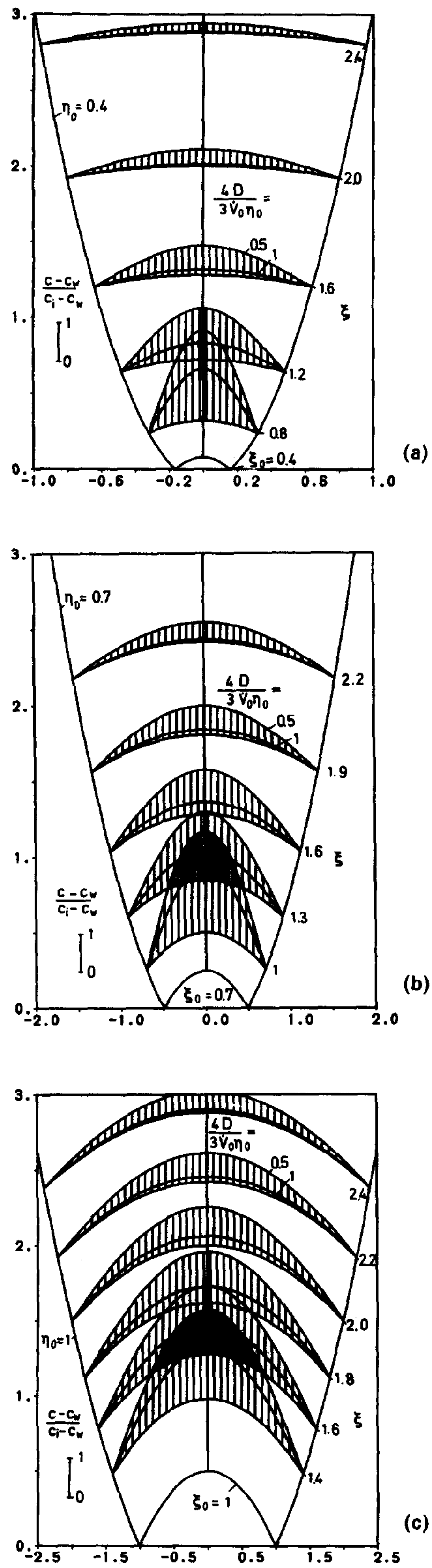


FIG. 3. Local concentration for constant wall and initial concentration.

ocity distribution is shown in Fig. 2(a) (see equation (12)). The velocity distribution is a function of both coordinates ξ and η and decreases along increasing ξ. The arrows indicate by their lengths the magnitude of the local velocity vector and show in addition the direction of the flow. The unity of the magnitude of $u(\xi, \eta)/\dot{V}_0$ [m^{-1}] is shown on the upper right of Fig. 2(a). The velocity decreases with increasing ξ-value and increasing η as well. For a wider conduit $\eta_0 = 0.7$ and 1.0 the velocity distribution per unit volumetric flow is presented in Figs. 2(b) and (c), in which the magnitude of $u(\xi, \eta)/\dot{V}_0$ is becoming smaller for larger η_0, i.e. conduit width. The local concentration (equation (19)) is presented in Figs. 3(a)–(c) for the same conduits $\eta_0 = 0.4$, 0.7 and 1.0. The initial concentration at the inlet $\xi = \xi_0$ was chosen to be of constant magnitude c_i across the conduit cross-section $\xi = \xi_0$. Figure 3(a) shows the local concentration $(c-c_w)/(c_i-c_w)$ for a slowly increasing cross-section of the conduit, i.e. for $\eta_0 = 0.4$. At the inlet $\xi_0 = 0.4$ the initial concentration was given as $c = c_i =$ constant. The magnitude of the concentration ratio $(c-c_w)/(c_i-c_w)$ is exhibited on the left-hand side of the graph. It may be noted that the magnitude of this ratio decreases with increasing ξ and η. Its magnitude, of course, depends on the magnitude of the parameter $4D/3\dot{V}_0\eta_0$, for which the values 0.5 and 1.0 have been used. An increase of the volumetric flow $\dot{V}_0$ renders a decrease of this parameter and therefore a larger concentration, while an increase of the diffusion coefficient D yields a smaller concentration. Figure 3(b) exhibits the local concentration for a wider parabolic conduit, i.e. $\eta_0 = 0.7$. The inlet was chosen to be at $\xi_0 = 0.7$. For an even wider conduit $\eta_0 = 1.0$ with the inlet at $\xi_0 = 1.0$ the results for the concentration are presented in Fig. 3(c). The upper curve is the local concentration for $4D/3\dot{V}_0\eta_0 = 0.5$, while the lower curves are the results for $4D/3\dot{V}_0\eta_0 = 1.0$.

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TRANSPORT DE MASSE DANS UN CONDUIT PARABOLIQUE

Résumé—La concentration locale est déterminée pour un liquide visqueux s'écoulant à l'intérieure d'un conduit parabolique. La concentration à la paroi est supposée constante, ainsi que la concentration initiale à l'entrée. L'écoulement liquide est supposé rampant et sa distribution de vitesse est déterminée en résolvant l'équation biharmonique de la fonction de courant. La concentration locale est évaluée numériquement à partir des résultats analytiques pour différents conduits paraboliques.

STOFFTRANSPORT IM PARABOLISCHEN KANAL

Zusammenfassung—Es wird die lokale Konzentration in einer viskosen Strömung durch einen parabolischen Kanal bei konstanter Wand- und Einlaßkonzentration bestimmt. Die kriechende Strömungsgeschwindigkeit wurde aus der Lösung der biharmonischen Differentialgleichung der Stromfunktion bestimmt. Die lokale Konzentration wurde aus den analytischen Ergebnissen für einige Parabolkanäle numerisch bestimmt.

ПЕРЕНОС МАССЫ В КАНАЛЕ ПАРАБОЛИЧЕСКОГО СЕЧЕНИЯ

Аннотация—Определена локальная концентрация вязкой жидкости при течении в канале параболического сечения. Концентрация жидкости на стенке канала и начальная концентрация на входе считаются постоянными. Течение происходит в ползущем режиме. Распределение скорости определяется решением бигармонического уравнения для функции тока, а локальная концентрация оценивается численно на основе аналитических результатов, полученных для различных параболических каналов.